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Optimality Criteria and Duality in Multiobjective Programming Involving Nonsmooth Invex Functions

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Abstract. In this paper a generalization of invexity is considered in a general form, by means of the concept of K-directional derivative. Then in the case of nonlinear multiobjective programming problems where the functions involved are nondifferentiable, we established sufficient optimality conditions without any convexity assumption of the K-directional derivative. Then we obtained some duality results.

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1. Introduction

In optimization theory convexity plays an important role in many aspects of Mathematical programming including sufficient optimality conditions and duality theorems. Several classes of functions have been defined for the purpose of weaking the limitations of convexity.

Hanson [8] and also Craven [5] introduced the class of invex functions. Furthermore Ben Israel and Mond [2] considered a class of functions called preinvex and also showed that the class of invex functions is equaivalent to the class of functions whose stationary point are global minima.

In [6] Craven has given Langragian necessary conditions for optimality, of both Fritz-John and Kuhn-Tucker types for a constrained minimization problem, where the functions are locally Lipshitz and the directional derivatives are assumed to have some convexity properties as functions of direction.

In [21], the concept of semilocally convex functions was extended to semilocally quasiconvex, semilocally pseduoconvex at a point with respect to a starshaped set and some optimally conditions and duality results for a multiobjective programming problems are obtained by taking the right differential of the objective and constraints at a point to be convex. Ortega and Rheinboldt [19] extended the concept of convex functions by defining arcwise connected functions on arcwise connected sets, for which points lying on continuous arcs, instead of line segments satisfy certain inequalities. Following this idea Aggarwal et al. [13] obtained Fritz-John type necessary optimality criteria for non-linear programs under the hypotheses that the right differentials, at the optimal point, of the objective and active constraint functions with respect to an arc are convex. In [2], some classes of functions was introduced and called preinvex functions. Then Antczack [1], by considering the concept of Pareto optimal solution, the Fritz-John type and Karush-Kuhn-Tucker necessary optimality conditions are stated under the assumptions that the directional derivatives have some pre-invexity property as functions of direction. Recently Preda [20] obtained necessary and sufficient optimality conditions for a nonlinear fractional multiple objective programming where the directional derivatives are assumed to have some generalized semilocally preinvex property. Also in [17] and [18] they stated necessary optimality conditions for a nondifferentiable multiobjective programming problems where the directional derivatives of objective functions and constraints are preinvex functions.

Parallel to the above development in nonsmooth multiobjective problems, there has been a very popular growth and applications of invexity theory to locally Lipchitz functions, with derivative replaced by the Clarke generalized gradient [4]. For example one can see, [9–12], and [16]. From the theoretical point of view, in this setting they using a particular local cone approximation.

We observe that most of the sufficiency results stated for nonsmooth invex functions are deduced from the necessary optimality conditions Fritz-John or Karush-kuhn-Tucker; hence they require the convexity or some generalized convexity assumptions of the directional derivatives or some regularity conditions.

In this paper we use definition of invexity for nonsmooth functions exploiting the concept of local cone approximation introduced in [7]. Moreover via such an approach, we give general sufficient optimality conditions for inequality constraint multiobjective problems without requiring the convexity of the directional derivatives or regularity conditions. In Section (2) we introduced generalized invexity along the lines of [3]. In Section (3) the optimality conditions are established. In Section (4) a number of duality theorems in the Mond–Wier type [23] as well as Wolf type dual [25] are established.

2. Preliminaries

Consider the following multiobjective programming problem (MP):

$$\begin{aligned} \min \quad & f(x) = (f_1(x), \dots, f_m(x)) \\ s.t. \quad & \\ & g_i(x) \leqslant 0, \quad i = 1, 2, \dots, p \\ & x \in X. \end{aligned}$$

where $f_i : X \to R, \quad g_i : X \to R$, and X is a nonempty open subset of R^n. Let $F_p = \{x \in X : g_i(x) \leqslant 0, \quad i \in P\}$ be the set of feasible solution for (MP) and denote $M = \{1, 2, \dots, m\}$, $P = \{1, 2, \dots, p\}$, and $I(\bar{x}) = \{i \in P : g_i(\bar{x}) = 0\}$.

Given a subset $A \subset R^n$ we will denote with clA and $intA$ respectively the topological closure and interior of A. If A is a convex cone, A^0 is its negative polar, that is

$$A^0 = \{x^* \in R^n | \langle x^*, x\rangle \leqslant 0, \forall x \in A\}.$$

Given the function $f : X \to R$, the epigraph of f is

$$epi\ f = \{(x, r) \in X \times R : f(x) \leqslant r\}.$$

The domain of f is the set dom f $= \{x \in R^n | f(x) < \infty\}$ and f is said proper if $f(x) > -\infty$ for any $x \in R^n$ and its domain is nonempty.

The set epi$f$ will be locally approximated at the point $(x, f(x))$ by a local cone approximation K and a positively homogenous function $f^K(x, .)$ will be uniquely determined.

DEFINITION 2.1. Let $f : X \to R, x \in X$ and K be a local cone approximation; the positively homogeneous function $f^K(x, .) : R^n \to [-\infty, \infty]$ defined by,

$$f^K(x; d) := \inf\{\xi \in R : (d, \xi) \in K(epif; (x, f(x)))\},$$

is called the K-directional derivative of f at x.

Now we introduce generalized directional derivatives used in literature;
– the upper Dini directional derivative of f at x

$$D_+ f(x, y) := \limsup_{t \to 0^+} \frac{f(x + ty) - f(x)}{t},$$

is associated to the cone of the feasible directions.

$$F(Q, x) := \{y \in R^n : \forall \{t_k\} \to 0^+, x + t_k y \in Q\};$$

– the lower directional derivative of f at x

$$D_{-}f(x,y):=\liminf_{t\to 0^{+}}\frac{f(x+ty)-f(x)}{t}$$

is associated to the cone of the weak feasible directions

$$WF(Q,x):=\{y\in R^{n}:\exists(t_k)\to 0^{+}\quad s.t.\quad x+t_k y\in Q\}.$$

– if f is locally Lipschitz, the Clarke directional derivative of f at x

$$f^{0}(x,y):=\limsup_{(x',t)\to(x,0^{+})}\frac{f(x'+ty)-f(x')}{t}$$

is associated to Clarke's tangent cone

$$T_c(Q,x):=\{y\in R^{n}:\forall\{x_k\}\to x\quad s.t.\quad x_k\in Q,\forall\{t_k\}\to 0^{+},\\ \exists\{y_k\}\to y\quad s.t.\quad x_k+t_k y_k\in Q\}.$$

DEFINITION 2.2. Let $f:X\to R$ and K be a local cone approximation, the function f is said to be K-subdifferentiable at x if there exists a convex compact set $\partial^{K}f(x)$ such that

$$f^{K}(x;y)=\max_{x^{*}\in\partial^{K}f(x)}\langle x^{*},y\rangle\quad\forall y\in R^{n},$$

the set $\partial^{K}f(x)$ is called the K-subdifferential of f at x.

DEFINITION 2.3. Let $f:X\to R$ and K be a local cone approximation, the function f is said to be K-invex at $\bar{x}$ if there exists a function $\eta:X\times X\to R^{n}$ such that

$$f(x)-f(\bar{x})\geqslant f^{K}(\bar{x},\eta(x,\bar{x})),\quad\forall x\in X.$$

The function η is said to be the kernel of the $K-$invexity.

REMARK 2.4. If f is K-subdifferentiable, then f is K-invex at $\bar{x}$ with respect to the kernel η if and only if for each $x\in X$

$$f(x)-f(\bar{x})\geqslant\langle\xi,\eta(x,\bar{x})\rangle,\quad\forall\xi\in\partial^{K}f(\bar{x}).$$

DEFINITION 2.5. Let K be a local cone approximation; the function $f:X\to R$ is said to be,

(i) K-pseudoinvex at $\bar{x}$ if there exists a function $\eta : X \times X \to R^n$ such that,

$$f^K(\bar{x}, \eta(x, \bar{x})) \geqslant 0 \Rightarrow f(x) \geqslant f(\bar{x}), \quad \forall x \in X;$$

(ii) K-quasiinvex at $\bar{x}$ if there exists a function $\eta : X \times X \to R^n$ such that,

$$f(x) \leqslant f(\bar{x}) \Rightarrow f^K(\bar{x}, \eta(x, \bar{x})) \leqslant 0, \quad \forall x \in X;$$

(iii) K-strictly quasiinvex at $\bar{x}$ if there exists a functional $\eta : X \times X \to R^n$ such that,

$$f(x) \leqslant f(\bar{x}) \Rightarrow f^K(x, \eta(x, \bar{x})) < 0, \quad \forall x \neq \bar{x} \in X.$$

3. Optimality Conditions

In this section we study the problem (MP).

$$\begin{array}{lll}(\text{MP}) & \min & f(x) = (f_1(x), \ldots, f_m(x)) \\ & & g_i(x) \leqslant 0, \quad i = 1, 2, \ldots, p \\ & & x \in X.\end{array}$$

DEFINITION 3.1. Let $\bar{x}$ be a feasible point for (MP) and K is local cone approximation; the point $\bar{x}$ is said to be

(i) strongly efficient stationary point for the problem (MP) with respect to K if the following system is impossible,

$$\begin{array}{ll}f_i^K(\bar{x}; d) < 0 & \text{for some} \quad i \in M, \\ f_i^K(\bar{x}; d) \leqslant 0 & \text{for all } i \in M, \\ g_j^K(\bar{x}; d) \leqslant 0 & \text{for all } j \in I(\bar{x}). \qquad (S_1)\end{array}$$

(ii) weakly efficient stationary point for the problem (MP) with respect to (K) if the following system is impossible,

$$\begin{array}{ll}f_i^K(\bar{x}, d) < 0 & \forall i \in M, \\ g_j^K(\bar{x}, d) < 0 & \forall j \in I(\bar{x}). \qquad (S_2)\end{array}$$

It is always possible to choose suitable local cone approximation K such that every efficient solution $\bar{x}$ is a weakly or strongly efficient stationary point with respect to K. For instance in [1], [24] and also in [17] it was shown that for $K = WF$, and $d = \eta(x, \bar{x})$ every efficient solution $\bar{x}$ is a weakly efficient stationary point. For differentiable functions we can see in [14], that every efficient solution for the problem (MP) under some constraint qualification is strongly efficient stationary point.

We will prove, under suitably assumptions of invexity, it is possible to deduce sufficient optimality condition directly from impossibility of the system (S_1) or (S_2).

THEOREM 3.2. *Let $\bar{x}$ be a strongly stationary efficient point for the problem (MP) with respect to K, if f_i's are K-invex at $\bar{x}$ and g_j's are K-quasiinvex at $\bar{x}$ with respect to the same kernel η for all functions, then $\bar{x}$ is an efficient solution for (MP).*

Proof. Let $\bar{x}$ its not efficient for (MP), then there exists $x \in F_p$ such that

$$f_i(x) < f_i(\bar{x}) \quad \text{for some } i \in M$$
$$f_j(x) \leqslant f_j(\bar{x}) \quad \text{for all } j \in M.$$

By K-invexity of f_i's, we have

$$f_i^K(\bar{x}, \eta(x, \bar{x})) < 0 \quad \text{for some } i \in M, \tag{1}$$

$$f_i^K(\bar{x}, \eta(x, \bar{x})) \leqslant 0 \quad \text{for all } i \in M. \tag{2}$$

Since $x \in F_p$, then $g_j(x) \leqslant g_j(\bar{x})$, $j \in I(\bar{x})$. By K-quasiinvexity of g_j's, we have

$$g_j^K(\bar{x}, \eta(x, \bar{x})) \leqslant 0, \quad \forall j \in I(\bar{x}). \tag{3}$$

□

which (1), (2) and (3) contradict to K-strongly stationary efficient point $\bar{x}$.

EXAMPLES 3.3. Given the problem

$$\begin{aligned} (\text{MP}) \quad \min \quad & f(x) = (f_1(x), f_2(x)) \\ & g_1(x) \leqslant 0 \\ & x \in R^2, \end{aligned}$$

where $f_1(x_1, x_2) = |x_1| - |x_2|$, $f_2(x_1, x_2) = 3|x_1| - |x_2|$ and $g_1(x_1, x_2) = 10|x_2| - 3|x_1|$.

If we consider local cone approximation $K = T_c$ and $x^* = (0, 0)$, then we obtain

$$f_1^K((0,0), (\eta_1, \eta_2)) = |\eta_1| - |\eta_2|$$
$$f_2^K((0,0), (\eta_1, \eta_2)) = 3|\eta_1| - |\eta_2|$$
$$g_1^K((0,0), (\eta_1, \eta_2)) = 10|\eta_2| - 3|\eta_1|.$$

It is easy to verify that $x^* = (0,0)$ is a strongly stationary efficient point for the problem (MP) with respect to K, that is the following system is impossible

$$f_i^K(x^*, \eta) < 0 \quad \text{for some } i \in M,$$
$$f_j^K(x^*; \eta) \leqslant 0, \quad \text{for all } j \in M = \{1, 2\}$$
$$g_1^K(x^*, \eta) \leqslant 0.$$

Since f_1 and f_2 are K-invex and g_1 is K-quasiinvex at x^* with respect to the same kernel η,

$$\eta((x_1, x_2), (y_1, y_2)) = (x_1 - y_1, x_2 - y_2)$$

the conditions of Theorem 3.2 are satisfied and x^* is an efficient solution for (MP).

THEOREM 3.4. *Let $\bar{x}$ be a weakly stationary efficient point for the problem (MP) with respect to K, if f_i's are K-pseudoinvex and g_j's are K-strictly quasiinvex at $\bar{x}$ with respect to the same kernel η for all functions, then $\bar{x}$ is an efficient solution for (MP).*

Proof. Since $g_I(\bar{x}) = 0$, and by assumption on g_i's, we have

$$g_j(x) \leqslant g_j(\bar{x}) = 0, \quad \forall j \in I(\bar{x}), \; \forall x \in F_P,$$

$$g_j^K(\bar{x}, \eta(x, \bar{x})) < 0, \quad \forall j \in I(\bar{x}).$$

By using the above inequality and the weakly stationary efficient property of $\bar{x}$, we have

$$f_i^K(\bar{x}, \eta(x, \bar{x})) \geqslant 0, \quad \forall i \in M,$$

as f_i's are K-pseudoinvex at $\bar{x}$, hence

$$f_i(x) \geqslant f_i(\bar{x}), \quad \forall i \in M, \; \forall x \in F_P.$$

This complete the proof. □

DEFINITION 3.5. We say that $\bar{x}$ is K-generalized efficient solution of (MP) if $\bar{x}$ is an efficient for (MP) and for all $i = \{1, \ldots, m\}$, $f_i^K(\bar{x}; y) \geqslant 0$ for any $y \in R^n$ with $f_j^K(\bar{x}; y) \leqslant 0, j \in \{1, \ldots, m\} \setminus \{i\}$ and $g_i^K(\bar{x}, y) \leqslant 0, i \in I(\bar{x})$, where $I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\}$.

THEOREM 3.6. *Let $\bar{x}$ be a K-generalized efficient solution for* (MP) *and K be a convex local cone approximation containing* 0. *Assume that $f_i^K(\bar{x};.), i=1,\ldots,p$ and $g_i^K(\bar{x},.), i\in I(\bar{x})$ are proper, and continuous on the interior of their domains and for each $i=1,2,\ldots,m$,*

$$int\ dom f_i^K(\bar{x},.)\cap\bigcap_{I(\bar{x})}\{y\in int\ dom\ g_i^K(\bar{x},.)|g_i^K(\bar{x},y)<0\}\neq\emptyset. \qquad (I)$$

then there exist $\lambda_i^*\geqslant 0$ for each $i\in I(\bar{x})$ and $\tau_i^*\geqslant 0, i=1,2,\ldots,m$ such that

$$0\in\sum_{i=1}^{m}(\tau_i^*\partial^K f_i)(\bar{x})+\sum_{I(\bar{x})}\lambda_i^*\partial^K g_i(\bar{x}).$$

Proof. Since $\bar{x}$ is a K-generalized efficient solution for (MP), therefor by Proposition 4.1 in [22], for each $i\in\{1,\ldots,p\}$

$$0\in cl(\partial^K f_i(\bar{x})+G_i^0),$$

where

$$G_i=\{y\in R^n|f_j^K(\bar{x},y)\leqslant 0, j\neq i, g_i^K(\bar{x},y)\leqslant 0\quad\forall\ i\in I(\bar{x})\}.$$

By assumption (I), Lemma 2.3 and Lemma 2.4 in [15], we have

$$G_i^0=\sum_{j\neq i}\bigcup_{\tau_j\geqslant 0}\partial^K(\tau_j f_j)(\bar{x})+\sum_{I(\bar{x})}\bigcup_{\lambda_i\geqslant 0}\partial^K(\lambda_i g_i)(\bar{x}).$$

Therefore,

$$0\in\partial^K f_i(\bar{x})+\sum_{j\neq i}\bigcup\partial^K(\tau_j f_j)(\bar{x})+\sum_{I(\bar{x})}\bigcup\partial^K(\lambda_i g_i)(\bar{x}).$$

Now by Theorem 3.7 in [7], we obtain

$$0\in\sum_{i=1}^{m}\tau_i^*\partial^K f_i(\bar{x})+\sum_{I(\bar{x})}\lambda_i^*\partial^K g_i(\bar{x}),$$

this complete the proof. □

4. Duality

We consider the following dual problem, which is in the Mond–Weir type [23]

$$
\begin{aligned}
(DMW) \quad \max \quad & f(u)=(f_1(u),\dots,f_m(u)) \\
s.t. \quad & 0\in\sum_{i=1}^{m}\tau_i\partial^K f_i(u)+\sum_{j=1}^{p}\lambda_j\partial^K g_j(u) \\
& \lambda_j g_j(u)\geqslant 0, \quad \lambda_j\geqslant 0, \quad j=1,\dots,p \\
& \sum_{i=1}^{m}\tau_i=1, \quad \tau_i\geqslant 0.
\end{aligned}
$$

Let F_D be the set of feasible solution for (DMW). We establish weak and strong duality results for dual problems with K-subdifferentiable functions.

THEOREM 4.1 *(weak duality). Let x be a feasible solution for* (*MP*) *and* (u,τ,λ) *be feasible for* (*DMW*). *Assume that the following condition holds:*

(a) $f_i(\cdot), i\in M$ are K-strictly quasiinvex, $\lambda_i g_i(\cdot)$, $i\in P$ are K-quasiinvex at (u), and $\tau_i>0$, then the following can not hold:

$$f_i(x)<f_i(u) \quad \textit{for some } i\in M \tag{4}$$

$$f_j(x)\leqslant f_j(u) \quad \textit{for all } j\in M. \tag{5}$$

Proof. For each feasible x of (MP) and each (u,τ,λ) for (DMW) we have,

$$\lambda_i g_i(x)-\lambda_i g_i(u)\leqslant 0, \quad \forall i\in I(u),$$

and since each $\lambda_i g_i(\cdot)$ is K-quasiinvex at u this implies

$$\langle\lambda_i\eta_i,\eta(x,u)\rangle\leqslant 0, \quad \forall\eta_i\in\partial g_i^K(u). \tag{6}$$

Now suppose that contrary to the result of the theorem that (4) and (5) hold, then we have

$$
\begin{aligned}
& f_i(x)<f_i(u) \quad \text{for some } i\in M \\
& f_j(x)\leqslant f_j(u) \quad \text{for all } j\in M.
\end{aligned}
$$

By hypothesis on f_i's and (4) and (5) imply,

$$\left\langle\sum_{i=1}^{m}\tau_i\xi_i,\eta(x,u)\right\rangle<0, \quad \forall\xi_i\in\partial^K f_i(u). \tag{7}$$

It follows from 7 and 6 that

$$\left\langle \sum_{i=1}^{m} \tau_i \xi_i + \sum_{I(u)} \lambda_j \eta_j, \eta(x,u) \right\rangle < 0 \tag{8}$$

which contradict the dual feasibility of (u, τ, λ). □

REMARK 4.2. The weak duality theorem also hold under the following assumptions:

(b) $f_i, i = 1, 2, \ldots, m$ are K-quasiinvex and $\lambda_i g_i(\cdot), i = 1, 2, \ldots, p$ are K-strictly quasiinvex.
(c) $f_i, i = 1, 2, \ldots, m$ are K-pseudoinvex, $\lambda_i g_i(\cdot), i = 1, 2, \ldots, p$ are K-quasiinvex, $\tau_i > 0$.

THEOREM 4.3 *(Strong Duality). Let $\bar{x}$ be an K-generalized efficient solution of* (MP) *and condition (I) is holds. Then there exist $\tau \in R^m$ and $\lambda \in R^p$ such that $(\bar{x}, \tau, \lambda)$ is feasible for (DMW) and $\lambda^t g(\bar{x}) = 0$. If also weak duality theorem 4.1 holds between (MP) and (DMW), then $(\bar{x}, \tau, \lambda)$ is efficient for (DMW).*

Proof. By Theorem 3.6 there exist $\tau \in R^m, \tau_i \geqslant 0$ and $\lambda \in R^p, \lambda_i \geqslant 0$, such that

$$0 \in \sum_{i=1}^{m} \tau_i \partial^K f_i(\bar{x}) + \sum_{I(\bar{x})} \lambda_i \partial^K g_i(\bar{x})$$

$$\tau_i \geqslant 0, \sum_{i=1}^{m} \tau_i = 1, \lambda_i \geqslant 0,$$

taking $\lambda_i = 0$, for $i \notin I(x^*)$, we have $\lambda_i g_i(\bar{x}) = 0$ for all $i \in P$. It follows that $(\bar{x}, \tau, \lambda)$ is feasible for (DMW). Next suppose that $(\bar{x}, \tau, \lambda)$ is not an efficient solution of (DMW), then there exist a point $(y^*, \tau^*, \lambda^*) \in F_D$ such that

$$f_i(\bar{x}) < f_i(y^*) \quad \text{for some } i \in M,$$
$$f_j(\bar{x}) \leqslant f_j(y^*) \quad \text{for all } j \in M.$$

which violates the weak duality Theorem 4.1. Hence $(\bar{x}, \tau, \lambda)$ is indeed an efficient solution of (DMW). □

We continuous our results on duality for (MP) in this section by considering a Wolf type dual [25] problem of (MP) and proving weak and strong duality theorems. We consider the following general Wolf type dual to (MP).

$$\begin{aligned} \max \quad & (f_1(u)+\lambda^t g(u), \ldots, f_m(u)+\lambda^t g(u)) \\ s.t. \quad & 0 \in \sum_{i=1}^{m} \tau_i \partial^K f_i(u) + \sum_{j=1}^{p} \lambda_j \partial^K g_j(u) \\ & \sum_{i=1}^{m} \tau_i = 1, \quad \tau_i \geqslant 0, \ i \in M \quad \lambda_j \geqslant 0, \ j \in P. \quad (WDM) \end{aligned}$$

Let F_{D2} denote the set of all feasible points of (WDM).

THEOREM 4.4 *(Weak Duality). Let x and (u, τ, λ) be feasible solution for (MP) and (WDM), respectively. If the following conditions hold:*

(a) $\tau_i > 0$, $f_i(.), i=1,2,\ldots m$ and $g_i(.), i=1,2,\ldots p$ are K-invex. Then the following can not hold:

$$f_i(x) < f_i(u) + \lambda^t g(u) \quad \textit{for some } i \in M \tag{9}$$

$$f_j(x) \leqslant f_j(u) + \lambda^t g(u) \quad \textit{for all } j \in M. \tag{10}$$

Proof. By the given hypothesis (a) and the feasibility of the (u, τ, λ) for (WDM), we have

$$\begin{aligned} & \sum_{i=1}^{m} \tau_i \left[f_i(x) - \left[f_i(u) + \sum_{i=1}^{p} \lambda_i g_i(u) \right] \right] \\ & \quad \geqslant \sum_{i=1}^{m} \tau_i f_i^K(u, \eta(x,u)) + \sum_{i=1}^{p} \lambda_i g_i^K(u, \eta(x,u)) - \sum_{i=1}^{p} \lambda_i g_i(x) \geqslant 0. \end{aligned}$$

Thus,

$$\sum_{i=1}^{m} \tau_i f_i(x) \geqslant \sum_{i=1}^{m} \tau_i \left[f_i(u) + \sum_{i=1}^{p} \lambda_i g_i(u) \right]. \tag{11}$$

On the other hand, suppose contrary to the result that (9) and (10) hold. Since x is feasible for (MP) and $\tau_i \geqslant 0$, then we have

$$\sum_{i=1}^{m} \tau_i f_i(x) < \sum_{i=1}^{m} \tau_i f_i(u) + \sum_{i=1}^{p} \lambda_i g_i(u),$$

which contradictions (11). □

THEOREM 4.5 (Strong Duality). *Let $\bar{x}$ be an K-generalized efficient solution of (MP) and condition (I) is holds. Then there exist $\tau \in R^m$ and $\lambda \in R^p$ such that $(\bar{x}, \tau, \lambda)$ is feasible for (MWD). If also weak duality theorem 4.4 holds between (MP) and (WDM), then $(\bar{x}, \tau, \lambda)$ is efficient for (WDM).*

Proof. By Theorem 3.6 there exist $\tau \in R^m, \tau_i \geqslant 0$ and $\lambda \in R^p, \lambda_i \geqslant 0$, such that

$$0 \in \sum_{i=1}^{m} \tau_i \partial^K f_i(\bar{x}) + \sum_{I(\bar{x})} \lambda_i \partial^K g_i(\bar{x})$$

$$\tau_i \geqslant 0, \sum_{i=1}^{m} \tau_i = 1, \lambda_i \geqslant 0,$$

taking $\lambda_i = 0$, for $i \notin I(x^*)$, it follows that $(\bar{x}, \tau, \lambda)$ is feasible for (WDM). Next suppose that $(\bar{x}, \tau, \lambda)$ is not an efficient solution of (WDM), then there exist a point $(x^*, \tau^*, \lambda^*) \in F_D$ such that

$$f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i g_i(\bar{x}) < f_i(x^*) + \sum_{i=1}^{p} \lambda_i g_i(x^*) \quad \text{for some } i \in M$$

$$f_j(\bar{x}) + \sum_{i=1}^{p} \lambda_i g_i(\bar{x}) \leqslant f_j(x^*) + \sum_{i=1}^{p} \lambda_i g_i(x^*) \quad \text{for all } j \in M. \qquad \Box$$

Since $\bar{x}$ is feasible then we have

$$f_i(\bar{x}) < f_i(x^*) + \sum_{i=1}^{p} \lambda_i g_i(x^*) \quad \text{for some } i \in M$$

$$f_i(\bar{x}) \leqslant f_i(x^*) + \sum_{i=1}^{p} \lambda_i g_i(x^*) \quad \text{for all } i \in M,$$

which violates the weak duality Theorem 4.4 Hence $(\bar{x}, \tau, \lambda)$ is indeed an efficient solution of (WDM).

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